Proceedings of the Jangjeon Mathematical Society
28 (2025), No. 4, pp. 675 – 690

www.jangjeon.or.kr
http://dx.doi.org/10.17777/pjms2025.28.4.020

EXISTENCE RESULTS FOR BOUNDARY VALUE PROBLEMS FOR NON-LINEAR FRACTIONAL INTEGRO DIFFERENTIAL INCLUSION WITH DISCRETE DELAY AND INTEGRAL CONDITIONS

BRIJENDRA KUMAR CHAURASIYA* AND AVADHESH KUMAR

Abstract. This study investigates non-linear fractional integro-differential inclusions, including discrete delay terms governed by integral conditions. The primary aim is to implement solution's existence for both convex as well as non-convex cases, which are crucial in many mathematical and physical models. The subsistence of at least one result is established by the use of Leray-Schauder fixed point theorem. Both convex and non-convex scenarios are carefully explored due to its impact in areas like control theory along with dynamic systems. Moreover, the topological structure of the problem is not just examined, but thoroughly scrutinized to validate the practical relevance of the solutions. This topological examination is key to understanding the complex behaviour of solutions in systems characterized by non-linearities and delays, providing a solid foundation for our findings. Overall, the study provides valuable theoretical insights that contribute to a more profound comprehension of fractional differential inclusions and its potential applications in real-world scenarios.

2010 Mathematics Subject Classification. 34B15, 34A60, 26A33.

Keywords and phrases. Caputo fractional derivative; differential inclusion; fractional integral; Boundary value problem; integral conditions.

1. Background

Fractional order differential equations become a critical tool in modelling complex real world phenomena across various domains of science and engineering. These equations, characterized by non-integer order derivatives, extend classical differential equations, offering advanced modelling techniques for systems in areas such as viscoelasticity [25], electro-chemistry [10], control systems [13], porous media [8], and electromagnetism [12] . The distinctive ability of fractional derivatives to capture memory effects and hereditary properties makes them highly valuable for modelling systems where traditional methods fall short.

Over the years, fractional calculus has driven substantial progress in the field of both ordinary differential equations as well as partial differential equations, particularly with the help of Caputo and Riemann-Liouville fractional derivatives [2]. Foundational work by Kilbas et al. [18]-[17], Miller and Ross [21], and later studies by Agarwal et al.[3], Benchohra et al. [4], and Benchohra and Hamani [5], have made important contribution in understanding the solution's existence and uniqueness for fractional differential

Date: Submitted on 14 October 2024.

equations [6]-[11]. Research such as has further extended these concepts by investigating initial value condition combined with various fractional-order functional differential equations added with the infinite delays [20]-[27].

Despite these advancements, a significant research gap remains regarding boundary value issues involving fractional differential inclusions combined with integral boundary conditions [9]-[23]. Most existing research has centred on classical boundary conditions or relatively simple cases, while more complex situations, such as multi-point or non-local boundary conditions, have received less attention. These types of problems are crucial in real-world applications, including community dynamics [19] and biological systems [30], where memory and delayed responses [32] play a significant role. The exploration of these problems, particularly in non-convex settings, remains underdeveloped in the literature [7]-[31].

Addressing this gap, the present study aims to explore boundary value issues combines fractional differential together with inclusions that involve discrete delay terms and integral boundary conditions. Specifically, we examine solution's existence for convex as well as non-convex cases, contributing to a deeper theoretical understanding of these issues. Based on the above information our proposed model is formulated as:

$$(1.1)\qquad \begin{cases} {}^cD^\delta \hbar(\mathsf{t}) \in \mathcal{H}\left(\mathsf{t}, \hbar(\mathsf{t}), \hbar(\mathsf{t}-\kappa), \hbar^{\mho}(\mathsf{t})\right), \delta \in (1,2], \mathsf{t} \in \jmath = [0, \dot{\psi}] \\ \hbar(0) - \hbar'(0) = \int_0^{\dot{\psi}} \kappa(s, \hbar) ds \\ \hbar(\dot{\psi}) + \hbar'(\dot{\psi}) = \int_0^{\dot{\psi}} \bar{\kappa}(s, \hbar) ds, \end{cases}$$

within this context, ${}^cD^\delta$ represents a type of fractional derivative called the Caputo derivative, while $\mathcal{H}$ denotes a set-valued operation that maps $\jmath \times \mathsf{B} \times \mathsf{B} \times \mathsf{B}$ to $\rho(\mathsf{B})$, where B represents the real number. The collection of all non-empty subclasses of B. The terms κ and $\bar{\kappa}$ are defined as continuous functions on $\jmath \times \mathsf{B}$, also $\hbar(\mathsf{t}-\kappa)$ deals with the delay term and κ represents discrete delay. Additionally, the integro term $\hbar^{\mho} = \int_0^{\dot{\psi}} \gamma(\dot{\psi}, s)\hbar(s) ds$ is integral to the system's dynamics.

By examining both convex and non-convex cases, this study fills an important gap in the literature, offering new theoretical insights into fractional differential inclusions conjunction with integral boundary conditions. Additionally, it emphasizes their practical applications in areas involving delayed and memory-dependent processes.

This paper is structured into several sections, each contributing to the exploration of our research area.

- Section 2 introduces the initial findings necessary for the subsequent sections.
- Section 3 presents a few existence results.
- Section 4 provides two results for non-convex valued right-hand sides.
- Section 5 discusses about the topological framework of the explanation set.
- Section 6 presents a pattern to clarify the discussed concepts.

These results expand upon findings from the previously cited literature to encompass the multivalued case, representing a novel improvement to this imminent area of inquiry.

2. Preparatories

This section introduces some basic notions and mathematical structures, along with the relevant symbols that are used for the analysis outlined in the current manuscript. Consider $\Delta(\jmath, \mathsf{B})$ is basically the Banach space of all criteria from $\jmath$ to B, equipped to the norm:

$$\|\hbar\|_\infty = \sup\{|\hbar(\mathsf{t})| : 0 \le \mathsf{t} \le \dot{\psi}\}.$$

Next, Assume $\mathcal{L}^1(\jmath, \mathsf{B})$ represent the complete norm linear space consisting Lebesgue integrable mappings $\hbar : \jmath \to \mathsf{B}$, with respect to the norm defined as:

$$\|\hbar\|_{\mathcal{L}^1} = \int_0^{\dot{\psi}} |\hbar(\mathsf{t})| d\mathsf{t}.$$

Similarly, $\mathcal{L}^\infty(\jmath, \mathsf{B})$ denotes the normed liner space consisting of all specifically bounded mappings $\hbar : \jmath \to \mathsf{B}$, in additional to the norm:

$$\|\hbar\|_{\mathcal{L}^\infty} = \inf\{l > 0 : |\hbar(\mathsf{t})| \le l, \forall\ \mathsf{t} \in \jmath\}.$$

Furthermore, $AC^1(\jmath, \mathsf{B})$ refers to the dimensions of absolutely continuous operation $\hbar : \jmath \to \mathsf{B}$, where their derivative $\hbar'$ is also continuous. Let $(\aleph, \|\cdot\|)$ be a complete norm space, and contemplate the following features:

- $-\theta_{cl} = \{\hbar \in \theta(\aleph) : \hbar \text{ is closed}\}$
- $-\theta_b = \{\hbar \in \theta(\aleph) : \hbar \text{ is bounded}\}$
- $-\theta_{cp} = \{\hbar \in \theta(\aleph) : \hbar \text{ is compact}\}$
- $-\theta_{cp,c} = \{\hbar \in \theta(\aleph) : \hbar \text{ is compact and convex}\}$.

A set-valued operator $\varpi : \aleph \to \theta(\aleph)$ is identified as convex regions if $\varpi(\hbar)$ is convex to every $\hbar \in \aleph$. The operator ϖ remains limited on finite subsets if for any $B \in \theta_b(\aleph)$, the set $\varpi(B) = \cup_{\hbar \in B} \varpi(\hbar)$ is bounded in $\aleph$, that signifies:

$$\sup_{\hbar \in B}\{\sup\{|u| : u \in \varpi(\hbar)\}\} < \infty.$$

The operator ϖ is referred as upper semi-continuous on $\aleph$ if for any given $\hbar_0 \in \aleph, \varpi(\hbar_0)$ is a compact subset of $\aleph$, and unbounded subset $\varnothing$ in $\aleph$ including $\varpi(\hbar_0)$, there is an open neighbourhood $\varnothing_0$ of $\hbar_0$ in such a way that $\varpi(\varnothing_0) \subset \varnothing$.

Additionally, ϖ is classified as completely continuous if for every $B \in \theta_b(\aleph), \varpi(B)$ is relatively compact. Furthermore, ϖ is upper semi-continuous under the condition that ϖ possesses a closed graph, meaning:

$$\hbar_n \to \hbar^*, \hbar_n \in \varpi(\hbar_n) \Longrightarrow \hbar^* \in \varpi(\hbar^*).$$

If ϖ is fully continuous with non-empty bounded and closed sets, ensuring the existence of a fixed point, means there is $\hbar \in \aleph$ in such a way that $\hbar \in \varpi(\hbar)$. The collection of solutions that remain unchanged under the multivalued mapping ϖ is signified by $\mathrm{Fix}(\varpi)$.

A set valued mapping $\varpi : \jmath \to \theta_{cl}(\mathsf{B})$ is implied for measurablility to entire $\hbar \in \mathsf{B}$, the operator:

$$\varpi^{-1}(\hbar) = \{x \in \jmath \mid \hbar \in \varpi(x)\}$$

is measurable.

Definition 2.1. [24] A set valued mapping $\mathcal{H} : \jmath \times \mathsf{B} \to \theta(\mathsf{B})$ is Carath$\acute{e}$odory if

- $\mathsf{t} \mapsto \mathcal{H}(\mathsf{t}, u)$ is measurable to all $u \in \mathsf{B}$,
- $u \mapsto \mathcal{H}(\mathsf{t}, u)$ is upper semi-continuous to every $\mathsf{t} \in \jmath$.

In this context:

- Measurability of $\mathsf{t} \mapsto \mathcal{H}(\mathsf{t}, u)$ means that for every fixed u, the function $\mathcal{H}(\mathsf{t}, u)$ behaves in a way compatible with the underlying measure space.
- Upper semi-continuity of $u \mapsto \mathcal{H}(\mathsf{t}, u)$ means that for almost every t, the map is continuous in u from above, meaning the values do not jump upwards abruptly.

For every $\hbar \in \Delta(\jmath, \mathsf{B})$, the cluster of selection is defined by

$$S_{\mathcal{H},\hbar} = \left\{v \in \mathcal{L}^1(\jmath, \mathsf{B}) : v(\mathsf{t}) \in \mathcal{H}\left(\mathsf{t}, \hbar(\mathsf{t}), \hbar(\mathsf{t}-\kappa), \hbar^{\hbar}(\mathsf{t})\right), \forall\, \mathsf{t} \in \jmath\right\}.$$

Suppose $(\aleph, d)$ express a metric space along with the norm $(\aleph, |\cdot|)$. Assume the metric $\mathcal{H}_d : \rho(\aleph) \times \rho(\aleph) \to \Re_+ \cup \{\infty\}$, defined as:

$$\mathcal{H}_d(R, G) = \max\left\{\sup_{p\in R} d(p, G), \sup_{q\in G} d(R, q)\right\},$$

at which $d(R, q) = \inf_{p\in R} d(p, q)$ and $d(p, G) = \inf_{p\in S} d(p, q)$. Then, $(\theta_{q,cl}, \mathcal{H}_d)$ is a metric space, and $(\theta_{cl}, \mathcal{H}_d)$ is a universal metric space [17].

Definition 2.2. [23] A set-valued map $\varnothing : \aleph \to \theta_{cl}$ referred to as:

- γ-Lipschitz if there is $\gamma > 0$ with characteristic,
$$\mathcal{H}_d(\varnothing(\mathsf{t}), \varnothing(\hbar)) \leq \gamma d(\mathsf{t}, \hbar), \quad \forall \mathsf{t}, \hbar \in \aleph.$$
- A mapping is called a contraction if it contains γ-Lipschitz with condition $\gamma < 1$.

Definition 2.3. [27] For a given a function $\bar{\kappa} \in \mathcal{L}^1([p, q], \Re_+)$ of fractional order $\delta \in \mathsf{B}_+$, is explained by:

$$I_p^\delta \bar{\kappa}(\mathsf{t}) = \frac{1}{\Gamma(\delta)} \int_p^{\mathsf{t}} (\mathsf{t} - s)^{\delta-1} \bar{\kappa}(s) ds$$

in which, the term Γ is the gamma operator If $p = 0$, this is expressed as $I^\delta \bar{\kappa}(\mathsf{t}) = \bar{\kappa}(\mathsf{t}) * \varphi_\delta(\mathsf{t})$, where $\varphi_\delta(\mathsf{t}) = \frac{\mathsf{t}^{\delta-1}}{\Gamma(\delta)}$ for $\mathsf{t} > 0$, and $\varphi_\delta(\mathsf{t}) = 0$ for $\mathsf{t} \leq 0$.

Definition 2.4. [27] The RL fractional δ-th order derivative for a function $\bar{\kappa} \in [p, q]$ is explained by:

$$\left(D_{p^+}^\delta \bar{\kappa}\right)(\mathsf{t}) = \frac{1}{\Gamma(n-\delta)} \frac{d^n}{d\mathsf{t}^n} \int_p^{\mathsf{t}} (\mathsf{t} - s)^{n-\delta-1} \bar{\kappa}(s) ds$$

where $n = [\delta] + 1$ and $[\delta]$ represents the integral part of δ.

Definition 2.5. [27] For given a function $\bar{\kappa} \in [p, q]$, the Caputo fractional δth order derivative is represented by:

$$\left({}^c D_{p^+}^\delta \bar{\kappa}\right)(\mathsf{t}) = \frac{1}{\Gamma(n-\delta)} \int_p^{\mathsf{t}} (\mathsf{t} - s)^{n-\delta-1} \bar{\kappa}^{(n)}(s) ds$$

at which $n = [\delta] + 1$.

3. The Convex Case

This section discusses the solution's existence on the behalf of the perimeter constraint problem (1.1) under the assumption that the the values on the right-hand side are convex. Our hypothesis begins by assuming that F is a compact map whose values are convex sets.

Proposition 3.1. *[34] An operation $\hbar \in AC^1(\jmath, \mathsf{B})$ is referred to satisfies the result of* (1.1) *if we have an operation $v \in \mathcal{L}^1(\jmath, \mathsf{B})$ in such a such way:*

$$v(\mathsf{t}) \in F\left(\mathsf{t}, \hbar(\mathsf{t}), \hbar(\mathsf{t}-\kappa), \hbar^{\mho}(\mathsf{t})\right), \quad \forall \mathsf{t} \in \jmath$$

and:

$$^cD^{\delta}\hbar(\mathsf{t}) = v(\mathsf{t}), \quad 1 < \delta \le 2, \quad \mathsf{t} \in \jmath$$

at which the operator $\hbar$ fulfils the associated boundary conditions in (1.1).

Lemma 3.2. *[34] Given $\delta > 0$, the associated solution for the system:*

$$^cD^{\delta}\bar{\kappa}(\mathsf{t}) = 0$$

is presented as:

$$\bar{\kappa}(\mathsf{t}) = l_0 + l_1\mathsf{t} + l_2\mathsf{t}^2 + \cdots + l_{n-1}\mathsf{t}^{n-1},$$

at which $l_i \in \mathsf{B}, i = 0, 1, 2, \ldots, n-1,$ and $n = [\delta] + 1$.

Lemma 3.3. *[34] For $\delta > 0$, we have:*

$$I^{\delta l}D^{\delta}\bar{\kappa}(\mathsf{t}) = \bar{\kappa}(\mathsf{t}) + l_0 + l_1\mathsf{t} + l_2\mathsf{t}^2 + \cdots + l_{n-1}\mathsf{t}^{n-1},$$

where $l_i \in \mathsf{B}, i = 0, 1, 2, \ldots, n-1,$ and $n = [\delta] + 1$.

Lemma 3.4. *Assume $1 < \delta \le 2$, along with $\sigma, \rho_1, \rho_2 : \jmath \to \mathsf{B}$ are continuous. A mapping $\hbar$ is a result for the following integral equation:*

$$\hbar(\mathsf{t}) = P(\mathsf{t}) + \int_0^{\dot{\psi}} \varpi(\mathsf{t}, s)\sigma(s)ds \tag{3.1}$$

under which

$$P(\mathsf{t}) = \frac{\dot{\psi} + 1 - \mathsf{t}}{\dot{\psi} + 2}\int_0^{\dot{\psi}} \rho_1(s)ds + \frac{\dot{\psi} + 1}{\dot{\psi} + 2}\int_0^{\dot{\psi}} \rho_2(s)ds \tag{3.2}$$

and

$$\varpi(\mathsf{t}, s) = \begin{cases} \frac{(\mathsf{t}-s)^{(\delta-1)}}{\Gamma(\delta)} - \frac{(1+\mathsf{t})(\mathsf{t}-s)^{(\delta-1)}}{(\dot{\psi}+2\Gamma(\delta)} - \frac{(1+\mathsf{t})(\dot{\psi}-s)^{(\delta-2)}}{(\dot{\psi}+2)\Gamma(\delta-1)}, & 0 \le s \le \mathsf{t} \\ -\frac{(1+\mathsf{t})(\mathsf{t}-s)}{(\dot{\psi}+2)\Gamma(\delta)} - \frac{(1+\mathsf{t})(\dot{\psi}-s)(\delta-2)}{(\dot{\psi}+2)\Gamma(\delta-1)}, & \mathsf{t} \le s < \dot{\psi} \end{cases} \tag{3.3}$$

The function $\hbar$ is valid for this equation only if it satisfies the fractional expression given by

$$^cD^{\delta}\hbar(\mathsf{t}) = \sigma(\mathsf{t}), \quad \mathsf{t} \in \jmath \tag{3.4}$$

together with boundary conditions:

$$\hbar(0) - \hbar'(0) = \int_0^{\dot{\psi}} \rho_1(s)ds, \tag{3.5}$$

$$\hbar(\dot{\psi}) + \hbar'(\dot{\psi}) = \int_0^{\dot{\psi}} \rho_2(s)ds. \tag{3.6}$$

Proof. Suppose $\hbar$ satisfies equation (3.4). From Lemma (3.3), it is provided as:

$$\hbar(\mathsf{t}) = \frac{1}{\Gamma(\delta)}\int_0^{\mathsf{t}} (\mathsf{t} - s)^{(\delta-1)}\sigma(s)ds + l_0 + l_1\mathsf{t} \tag{3.7}$$

Differentiating, we get

$$\hbar'(\mathsf{t}) = \frac{1}{\Gamma(\delta - 1)}\int_0^{\mathsf{t}} (\mathsf{t} - s)^{(\delta-2)}\sigma(s)ds + l_1 \tag{3.8}$$

Substituting into equations (3.5) and (3.6), we get

$$l_0 - l_1 = \int_0^{\dot{\psi}} \rho_1(s)ds \tag{3.9}$$

and:

$$l_0 + l_1(\dot{\psi}+1) + \frac{1}{\Gamma(\delta)}\int_0^{\mathsf{t}} (\mathsf{t}-s)^{(\delta-1)}\sigma(s)ds + \frac{1}{\Gamma(\delta-1)}\int_0^{\mathsf{t}} (\mathsf{t}-s)^{(\delta-2)}\sigma(s)ds = \int_0^{\dot{\psi}} \rho_2(s)ds \tag{3.10}$$

Solving these equations gives the values of l_0 and l_1. Substituting them into equation (3.7), we get the required value for $\hbar(\mathsf{t})$. On the other hand, if $\hbar$ solves the equation (3.1), also it satisfies (3.4)-(3.6). $\square$

Theorem 3.5. *[3] Our hypothesis is derived from the non-linear alternative of the multivalued maps from the Leray-Schauder type. Suppose the upcoming assumptions hold:*
(A1): $\mathcal{H} : \dot{\psi} \times \Re \times \Re \times \Re \to \theta_{cp,c}(\mathsf{B})$ *behaves as a Carathėodory multivalued mapping.*
(A2): *There is* $p \in \mathcal{L}^{\infty}(\jmath, \mathsf{B}^{+})$ *combined with continuity and non-decreasing mapping* $\dot{\psi} : [0,\infty) \to (0,\infty)$ *in a way :*

$$\|\mathcal{H}(\mathsf{t},u)\|_p = \sup\{|v| : v \in \mathcal{H}(\mathsf{t},u)\} \leq p(\mathsf{t})\dot{\psi}(|u|), \quad \forall \mathsf{t} \in \jmath, u \in \mathsf{B}$$

(A3): *This notation* $\phi_{\kappa} \in \mathcal{L}^1(\jmath, \mathsf{B}^{+})$ *along with a continuous, non-decreasing mapping* $\dot{\psi}^* : [0,\infty) \to (0,\infty)$ *in a way:*

$$\|\kappa(\mathsf{t},u)\| \leq \phi_{\kappa}(\mathsf{t})\dot{\psi}^*(|u|), \quad \forall \mathsf{t} \in \jmath, u \in \mathsf{B}$$

(A4): *There is* $\phi_{\bar{\kappa}} \in \mathcal{L}^1(\jmath, \mathsf{B}^{+})$ *along a continuous, non-decreasing mapping* $\bar{\dot{\psi}} : [0,\infty) \to (0,\infty)$ *in a way:*

$$\|\bar{\kappa}(\mathsf{t},u)\| \leq \phi_{\bar{\kappa}}(\mathsf{t})\bar{\dot{\psi}}(|u|), \quad \forall \mathsf{t} \in \jmath, u \in \mathsf{B}$$

(A5): *There is* $l \in \mathcal{L}^{\infty}(\jmath, \mathsf{B}^{+})$ *in a way:*

$$\mathcal{H}_d(\mathcal{H}(\mathsf{t},u), \mathcal{H}(\mathsf{t},\bar{u})) \leq l(\mathsf{t})|u-\bar{u}|, \quad \forall u, \bar{u} \in \Re$$

and:

$$d(0, \mathcal{H}(\mathsf{t},0)) \leq l(\mathsf{t}), \quad \forall \mathsf{t} \in \jmath$$

(A6): *There is a constant* $\wp \geq 0$ *in such a way:*

$$\frac{\wp}{a\dot{\psi}^*(\wp) + b\bar{\dot{\psi}}(\wp) + c\varpi\dot{\psi}(\wp)} > 1 \tag{3.11}$$

where:

$$a = \tfrac{\dot{\psi}+1}{\dot{\psi}+2}\int_0^{\dot{\psi}} \phi_{\kappa}(s)ds, \quad b = \tfrac{\dot{\psi}+1}{\dot{\psi}+2}\int_0^{\dot{\psi}} \phi_{\bar{\kappa}}(s)ds, \quad c = \|\theta\|_{\mathcal{L}^{\infty}}$$

Then the system (1.1) *has at least one solution on* $\jmath$.

Proof. Our aim is to reformulate system (1.1) to a fixed-point issue by defining a set valued map given by the following equation

$$N(\hbar) = \left\{ \bar{\kappa} \in C(\jmath, \mathsf{B}) : \bar{\kappa}(\mathsf{t}) = P_{\hbar}(\mathsf{t}) + \int_0^{\dot{\psi}} \varpi(\mathsf{t},s)v(s)ds, v \in S_{\mathcal{H},\hbar} \right\} \tag{3.12}$$

along with the term

$$P_{\hbar}(\mathsf{t}) = \frac{\dot{\psi}+1-\mathsf{t}}{\dot{\psi}+2}\int_0^{\mathsf{t}} \kappa(s,\hbar(s))ds + \frac{\dot{\psi}+1}{\dot{\psi}+2}\int_0^{\mathsf{t}} \bar{\kappa}(s,\hbar(s))ds \tag{3.13}$$

and the term $\varpi(\mathsf{t},s)$ is already expressed in equation (3.3) and with the help of Lemma (3.4), we can say that the fixed points of $\varnothing$ are outcomes to (1.1).

The proof proceeds through numerous stages:
Step 1. To demonstrate that $\varnothing(\hbar)$ to be convex for every single $\hbar \in \Delta(\jmath, \mathsf{B})$. Assume that $\bar{\kappa}_1, \bar{\kappa}_2 \in \varnothing(\hbar)$, and suppose there is $v_1, v_2 \in S_{\mathcal{H},\hbar}$ in such a way that to all $\mathsf{t} \in \jmath$,

$$\bar{\kappa}_i(\mathsf{t}) = P_\hbar(\mathsf{t}) + \int_0^{\dot{\psi}} \varpi(\mathsf{t}, s) v_i(s) ds, \quad i = 1, 2$$

For any $0 \leq d \leq 1$, define:

$$(d\bar{\kappa}_1 + (1-d)\bar{\kappa}_2)(\mathsf{t}) = P_\hbar(\mathsf{t}) + \int_0^{\dot{\psi}} \varpi(\mathsf{t}, s) \left[d v_1(s) + (1-d) v_2(s)\right] ds$$

Moreover, $S_{\mathcal{H},\hbar}$ is convex, we conclude that $d\bar{\kappa}_1 + (1-d)\bar{\kappa}_2 \in \varnothing(\hbar)$.
Step 2. Suppose $B_{\eta^*} = \{\hbar \in \Delta(\jmath, \mathsf{B}) : \|\hbar\|_\infty \leq \eta^*\}$ be enclosed within limits in $C(\jmath, \mathsf{B})$, and suppose that $\hbar \in B_{\eta^*}$, then to every $f \in \varnothing(\hbar)$ and $\mathsf{t} \in \jmath$, using conditions (A2) $-$ (A4), we obtain,

$$\begin{aligned}
|\bar{\kappa}(\mathsf{t})| \leq & \frac{\dot{\psi}+1}{\dot{\psi}+2} \int_0^{\dot{\psi}} |\kappa(s, \hbar(s))| ds + \frac{\dot{\psi}+1}{\dot{\psi}+2} \int_0^{\dot{\psi}} |\bar{\kappa}(s, \hbar(s))| ds \\
& + \int_0^{\dot{\psi}} \varpi(\mathsf{t}, s) |v(s)| ds \\
\leq & \frac{\dot{\psi}+1}{\dot{\psi}+2} \dot{\psi}^* (\|\hbar\|_\infty) \int_0^{\dot{\psi}} \phi_\kappa(s) ds + \frac{\dot{\psi}+1}{\dot{\psi}+2} \bar{\dot{\psi}} (\|\hbar\|_\infty) \int_0^{\dot{\psi}} \phi_{\bar{\kappa}}(s) ds \\
& + \dot{\psi} (\|\hbar\|_\infty) \|p\|_{\mathcal{L}^\infty} \bar{\varpi}
\end{aligned}$$

Thus

$$\|\bar{\kappa}(\mathsf{t})\| \leq \tfrac{\dot{\psi}+1}{\dot{\psi}+2} \dot{\psi}^* (\eta^*) \int_0^{\dot{\psi}} \phi_\kappa(s) ds + \tfrac{\dot{\psi}+1}{\dot{\psi}+2} \bar{\dot{\psi}} (\eta^*) \int_0^{\dot{\psi}} \phi_{\bar{\kappa}}(s) ds + \dot{\psi} (\eta^*) \|p\|_{\mathcal{L}^\infty \bar{\varpi}} := l$$

Step 3. For $\mathsf{t}_1, \mathsf{t}_2 \in \jmath$ with $\mathsf{t}_1 < \mathsf{t}_2$, and $\hbar \in B_{\eta^*}$, estimate:

$$\begin{aligned}
|\bar{\kappa}(\mathsf{t}_2) - \bar{\kappa}(\mathsf{t}_1)| \leq & \frac{\mathsf{t}_2 - \mathsf{t}_1}{\dot{\psi}+2} \int_0^{\dot{\psi}} |\kappa(s, \hbar(s))| ds + \frac{\mathsf{t}_2 - \mathsf{t}_1}{\dot{\psi}+2} \int_0^{\dot{\psi}} |\bar{\kappa}(s, \hbar(s))| ds \\
& + \int_0^{\dot{\psi}} |\varpi(\mathsf{t}_2, s) - \varpi(\mathsf{t}_1, s)| |v(s)| ds \\
\leq & \frac{\mathsf{t}_2 - \mathsf{t}_1}{\dot{\psi}+2} \dot{\psi}^* (\eta^*) \int_0^{\dot{\psi}} \phi_\kappa(s) ds + \frac{\mathsf{t}_2 - \mathsf{t}_1}{\dot{\psi}+2} \bar{\dot{\psi}} (\eta^*) \int_0^{\dot{\psi}} \phi_{\bar{\kappa}}(s) ds \\
& + \dot{\psi} (\eta^*) \|p\|_{\mathcal{L}^\infty} \int_0^{\dot{\psi}} |\varpi(\mathsf{t}_2, s) - \varpi(\mathsf{t}_1, s)| ds
\end{aligned}$$

the right-hand approaches to zero as $\mathsf{t}_1 \to \mathsf{t}_2$. Hence, as a consequence of the Arzela-Ascoli theorem, $\varnothing$ is completely continuous.

Step 4. Suppose $\hbar_n \to \hbar_*, \bar{\kappa}_n \in \varnothing(\hbar_n)$, along with $\bar{\kappa}_n \to \bar{\kappa}_*$. our aim is to establish $\bar{\kappa}_*$ in $\varnothing(\hbar_*)$. Since $\bar{\kappa}_n \in \varnothing(\hbar_n)$, there exist $v_n \in S_{\mathcal{H},\hbar_n}$ in a way so that for every $\mathsf{t} \in \jmath$

$$\bar{\kappa}_n(\mathsf{t}) = P_{\hbar_n}(\mathsf{t}) + \int_0^{\dot{\psi}} \varpi(\mathsf{t}, s) v_n(s) ds$$

We need to demonstrate that there is $v_* \in S_{\mathcal{H},\hbar_*}$ with

$$\bar{\kappa}_*(\mathsf{t}) = P_{\hbar_*}(\mathsf{t}) + \int_0^{\dot{\psi}} \varpi(\mathsf{t}, s) v_*(s) ds$$

Duo to the fact that $\mathcal{H}(\mathsf{t},\cdot)$ is upper semi-continuous and to all $\epsilon>0$, there is $n_0(\epsilon)\geq 0$ in a way that to each instance of $n\geq n_0$:

$$v_n(\mathsf{t})\in\mathcal{H}\left(\mathsf{t},\hbar_n(\mathsf{t})\right)\subset\mathcal{H}\left(\mathsf{t},\hbar_*(\mathsf{t})\right)+\epsilon B(0,1),\quad\forall\,\mathsf{t}\in\jmath$$

Duo to the fact that $\mathcal{H}$ has bounded as well as closed values, there is a subsequence v_{n_m} in such away that $v_{n_m}\to v_*$. Hence, $v_*(\mathsf{t})\in\mathcal{H}\left(\mathsf{t},\hbar_*(\mathsf{t})\right)$ for every $\mathsf{t}\in\jmath$, and we can show:

$$|\bar{\kappa}_{n_m}(\mathsf{t})-\bar{\kappa}_*(\mathsf{t})|\to 0,\quad\text{as}\quad m\to\infty$$

Step 5. Assume that $\hbar$ provide a solution for the given problem (1.1) and there is $v\in S_{\mathcal{H},\hbar}$ in such a way that for every $\mathsf{t}\in\jmath$

$$\begin{aligned}|\hbar(\mathsf{t})|\leq&\frac{\dot{\psi}+1}{\dot{\psi}+2}\int_0^{\dot{\psi}}\phi_\kappa(s)\dot{\psi}^*(|\hbar(s)|)ds+\frac{\dot{\psi}+1}{\dot{\psi}+2}\int_0^{\dot{\psi}}\phi_{\bar{\kappa}}(s)\bar{\dot{\psi}}(|\hbar(s)|)ds\\&+\int_0^{\dot{\psi}}\varpi(\mathsf{t},s)p(s)\dot{\psi}(|\hbar(s)|)ds\\\leq&\frac{\dot{\psi}+1}{\dot{\psi}+2}\dot{\psi}^*\left(\|\hbar\|_\infty\right)\int_0^{\dot{\psi}}\phi_\kappa(s)ds+\frac{\dot{\psi}+1}{\dot{\psi}+2}\bar{\dot{\psi}}\left(\|\hbar\|_\infty\right)\int_0^{\dot{\psi}}\phi_{\bar{\kappa}}(s)ds\\&+\dot{\psi}\left(\|\hbar\|_\infty\right)\bar{\varpi}\|p\|_{\mathcal{L}^\infty}.\end{aligned}$$

Thus,

$$\frac{\|\hbar\|_\infty}{a\dot{\psi}^*(\|\hbar\|_\infty)+b\bar{\dot{\psi}}(\|\hbar\|_\infty)+c\overline{\varpi}\dot{\psi}(\|\hbar\|_\infty)}\leq 1$$

Hence, a constant can be found $\wp$ under the condition that $\|\hbar\|_\infty\leq\wp$. Suppose

$$U=\left\{\hbar\in\Delta(\jmath,\mathsf{B}):\|\hbar\|_\infty<\wp\right\}.$$

Since, mapping $\varnothing:\bar{U}\to\theta(\Delta(\jmath,\mathsf{B}))$ is both completely as well as upper semi-continuous. Since no element $\hbar\in\partial U$ satisfies the condition $\hbar=\lambda\varnothing(\hbar)$ for some $\lambda\in(0,1)$, the non-linear alternative of the Leray-Schauder type states to a fixed point in $\hbar$ for $\bar{U}$, which solves the problem stated in (1.1). $\square$

4. The non convex case

This section of the paper discusses the existence of solutions for equation (1.1) in circumstances where the right-hand side is not convex. The fixed-point theorem is what the first result relies on, concerning contraction multivalued mappings as introduced by Covits and Nadler [7]. The second result relies on Bressan and Colombo's selection theorem [24], which addresses lower semi-continuous operators with reducible parts, together with the non-linear Leray-Schauder alternative. The transition to non-convex scenarios introduces challenges such as the lack of compactness, discontinuities in selections, and difficulty ensuring measurability and boundedness of mappings. To address these, assumptions (A7), (A8), and (A9) impose Lipschitz continuity and boundedness conditions, preparing the groundwork for applying Bressan and Colombo's theorem. This theorem enables the construction of measurable selections from multivalued mappings, ensuring the reducibility of the operator and facilitating the use of fixed-point arguments. Furthermore, it establishes the critical link between weak and strong convergence in $\mathcal{L}^1_w$, which is essential for proving the existence of solutions in non-convex settings. By overcoming the complexities of non-convexity, this approach ensures the analytical consistency required for the validity of theorem (4.1).

Theorem 4.1. *Consider the condition (A5) with subsequent hypotheses are satisfied:*
(A7) *A constant κ^* that is greater than zero exists, to ensure that*

$$|\kappa(\mathsf{t}, u) - \kappa(\mathsf{t}, \bar{u})| \leq \kappa^*|u - \bar{u}| \quad \text{to all values of } \mathsf{t} \in \jmath \text{ with } u, \bar{u} \in \mathsf{B}$$

(A8) *A constant κ^{**} that is greater than zero exists, to ensure that*

$$|\bar{\kappa}(\mathsf{t}, u) - \bar{\kappa}(\mathsf{t}, \bar{u})| \leq \kappa^{**}|u - \bar{u}| \quad \text{to all values of } \mathsf{t} \in \jmath \text{ and } u, \bar{u} \in \Re$$

(A9) *The multivalued map $\mathcal{H} : \jmath \times \mathsf{B} \to \theta_{cp}(\mathsf{B})$ fulfils the attributes that $\mathcal{H}(\cdot, u) : \jmath \to \theta_{cp}(\mathsf{B})$ is measurable and integrably bounded to all instances of $u \in \mathsf{B}$, provided that*

$$\left[\frac{\dot{\psi}(\dot{\psi}+1)}{\dot{\psi}+2}\kappa^* + \frac{\dot{\psi}(\dot{\psi}+1)}{\dot{\psi}+2}\kappa^{**} + \kappa\bar{\varpi}\right] < 1 \tag{4.1}$$

at which $\kappa = \|l\|_{\mathcal{L}^\infty}$, then equation (1.1) *contains one or more results on $\jmath$.*

Proof. Our aim is to demonstrate that for the given operator $\varnothing$ given in equation (3.12), the hypotheses of lemma (2.3) are satisfied. We will divide the analysis into two part.
Part 1: As $\varnothing(\hbar) \in \theta_{cl}(\Delta(\jmath, \mathsf{B}))$ for all $\hbar \in \Delta(\jmath, \mathsf{B})$ and suppose $(\bar{\kappa}_n)_{n\geq 0} \in \varnothing(\hbar)$ in such a way that $\bar{\kappa}_n \to \bar{\kappa} \in \Delta(\jmath, \mathsf{B})$. Then there is $v_n \in S_{\mathcal{H},\hbar}$ in such a way that, to all $\mathsf{t} \in \jmath$,

$$\bar{\kappa}_n(\mathsf{t}) = P_\hbar(\mathsf{t}) + \textstyle\int_0^{\dot{\psi}} \varpi(\mathsf{t}, s)v_n(s)ds.$$

From (A5) and the boundedness of the values of $\mathcal{H}$, we may select a subsequence if required, and deduce that v_n converges weakly to v in $\mathcal{L}^1_w(\jmath, \mathsf{B})$, a space whose topology is weak. By employing standard techniques for proving weak convergence implies strong convergence, It can be demonstrated that v_n converges strongly to v, with this we conclude that $v \in S_{\mathcal{H},\hbar}$. Therefore, to every $\mathsf{t} \in \jmath$,

$$\bar{\kappa}_n(\mathsf{t}) \to \bar{\kappa}(\mathsf{t}) = P_\hbar(\mathsf{t}) + \textstyle\int_0^{\dot{\psi}} \varpi(\mathsf{t}, s)v(s)ds,$$

thus $\bar{\kappa} \in N(\hbar)$.
Part 2: Consider that for $\gamma < 1$ in a way that

$$\mathcal{H}_d(\varnothing(\hbar), \varnothing(\bar{\hbar})) \leq \gamma\|\hbar - \bar{\hbar}\|_\infty \quad \text{for all } \hbar, \bar{\hbar} \in \Delta(\jmath, \mathsf{B})$$

Assume $\hbar, \bar{\hbar} \in \Delta(\jmath, \mathsf{B})$ and $\bar{\kappa}_1 \in \varnothing(\hbar)$. Then there is $v_1(\mathsf{t}) \in \mathcal{H}(\mathsf{t}, \hbar(\mathsf{t}))$ in such a way that, for every $\mathsf{t} \in \jmath$,

$$\bar{\kappa}_1(\mathsf{t}) = P_\hbar(\mathsf{t}) + \textstyle\int_0^{\dot{\psi}} \varpi(\mathsf{t}, s)v_1(s)ds$$

Using (A5), this implies that

$$\mathcal{H}_d(\mathcal{H}(\mathsf{t}, \hbar(\mathsf{t})), \mathcal{H}(\mathsf{t}, \bar{\hbar}(\mathsf{t}))) \leq l(\mathsf{t})|\hbar(\mathsf{t}) - \bar{\hbar}(\mathsf{t})|$$

Therefore, there is $\omega \in \mathcal{H}(\mathsf{t}, \bar{\hbar}(\mathsf{t}))$ in such a way that

$$|v_1(\mathsf{t}) - \omega| \leq l(\mathsf{t})|\hbar(\mathsf{t}) - \bar{\hbar}(\mathsf{t})|, \quad \mathsf{t} \in \jmath$$

Define a mapping $U : \jmath \to \theta(\mathsf{B})$ by

$$U(\mathsf{t}) = \left\{\Lambda \in \mathsf{B} : |v_1(\mathsf{t}) - \omega| \leq l(\mathsf{t})|\hbar(\mathsf{t}) - \bar{\hbar}(\mathsf{t})|\right\}$$

In view of the fact the set valued map $V(\mathsf{t}) = U(\mathsf{t}) \cap \mathcal{H}(\mathsf{t}, \bar{\hbar}(\mathsf{t}))$ is measurable, there is an operation $v_2(\mathsf{t})$, a measurable selection for V, in such a way that $v_2(\mathsf{t}) \in \mathcal{H}(\mathsf{t}, \bar{\hbar}(\mathsf{t}))$ with

$$|v_1(\mathsf{t}) - v_2(\mathsf{t})| \leq l(\mathsf{t})|\hbar(\mathsf{t}) - \bar{\hbar}(\mathsf{t})|, \quad \mathsf{t} \in \jmath$$

Considering each $\mathsf{t} \in \jmath$, define

$$\bar{\kappa}_2(\mathsf{t}) = P_{\bar{\hbar}}(\mathsf{t}) + \int_0^{\dot{\psi}} \varpi(\mathsf{t}, s) v_2(s) ds$$

in this case

$$P_{\bar{\hbar}}(\mathsf{t}) = \tfrac{\dot{\psi}+1-\mathsf{t}}{\dot{\psi}+2} \int_0^{\mathsf{t}} \kappa(s, \bar{\hbar}(s)) ds + \tfrac{\dot{\psi}+1}{\dot{\psi}+2} \int_0^{\mathsf{t}} \bar{\kappa}(s, \bar{\hbar}(s)) ds$$

Now, given $\mathsf{t} \in \jmath$,

$$\begin{aligned}
|\bar{\kappa}_1(\mathsf{t}) - \bar{\kappa}_2(\mathsf{t})| \leq & \frac{\dot{\psi}+1}{\dot{\psi}+2} \int_0^{\mathsf{t}} |\kappa(s, \hbar(s)) - \kappa(s, \bar{\hbar}(s))| ds \\
& + \frac{\dot{\psi}+1}{\dot{\psi}+2} \int_0^{\mathsf{t}} \left| \bar{\kappa}(s, \hbar(s)) - \bar{\kappa}(s), \bar{\hbar}(s) \right) | \, ds \\
& + \int_0^{\dot{\psi}} \varpi(\mathsf{t}, s) \, |v_1(s) - v_2(s)| \, ds \\
\leq & \frac{\dot{\psi}(\dot{\psi}+1)}{\dot{\psi}+2} \kappa^* \|\hbar - \bar{\hbar}\|_\infty + \frac{\dot{\psi}(\dot{\psi}+1)}{\dot{\psi}+2} \kappa^{**} \|\hbar - \bar{\hbar}\|_\infty \\
& + \bar{\varpi}\kappa \|\hbar - \bar{\hbar}\|_\infty
\end{aligned}$$

Thus,

$$\|\bar{\kappa}_1 - \bar{\kappa}_2\|_\infty \leq \left[\tfrac{\dot{\psi}(\dot{\psi}+1)}{\dot{\psi}+2} \kappa^* + \tfrac{\dot{\psi}(\dot{\psi}+1)}{\dot{\psi}+2} \kappa^{**} + \bar{\varpi}\kappa\right] \|\hbar - \bar{\hbar}\|_\infty,$$

Through a similar reasoning by swapping the position of $\hbar$ and $\bar{\hbar}$, it can be concluded that

$$\mathcal{H}_d(\varnothing(\hbar), \varnothing(\bar{\hbar})) \leq \left[\tfrac{\dot{\psi}(\dot{\psi}+1)}{\dot{\psi}+2} \kappa^* + \tfrac{\dot{\psi}(\dot{\psi}+1)}{\dot{\psi}+2} \kappa^{**} + \bar{\varpi}\kappa\right] \|\hbar - \bar{\hbar}\|_\infty$$

Thus, to begin with, due to condition (4.1), $\varnothing$ is a contraction mapping, and by Lemma (2.3), it contains a fixed point $\hbar$, which is one of the solutions to equation (1.1). Hence, that concludes the proof. □

Next, we show the solution for equation (1.1), motivated by the non-linear alternative of the Leray-Schauder kind in association with unique-valued mappings.

Definition 4.2. [25] Suppose $\hbar$ is a metric space with a countable dense subset (separable) and $\varnothing : \hbar \to \theta\left(\mathcal{L}^1([0, \dot{\psi}], \Re)\right)$ is a set valued map. We can say that $\varnothing$ contains certain properties if:

- $\varnothing$ exhibits lower semi-continuity;
- $\varnothing$ has non-empty, closed, and decomposable sets as values.

Suppose that $\mathcal{H} : [0, \dot{\psi}] \times \Re \to \theta(\mathsf{B})$ is a set valued mapping with non-empty bounded as well as closed values assigned to $\mathcal{H}$, the multi- function mapping $\mathcal{H} : \Delta([0, \dot{\psi}], \mathsf{B}) \to \theta\left(\mathcal{L}^1([0, \dot{\psi}], \mathsf{B})\right)$, is given by

$$\mathcal{H}(\hbar) = \left\{ w \in \theta\left(\mathcal{L}^1([0, \dot{\psi}], \mathsf{B})\right) : w(t) \in \mathcal{H}(\mathsf{t}, \hbar(\mathsf{t})) \text{ for all } \mathsf{t} \in [0, \dot{\psi}] \right\}$$

This mapping $\mathcal{H}$ is called the Niemytzki operator affiliated with $\mathcal{H}$.

Definition 4.3. Let us assume that $\mathcal{H} : [0, \dot{\psi}] \times \mathsf{B} \to \theta(\mathsf{B})$ be a set-valued mapping having nonempty closed as well as bounded values. Then, we declare $\mathcal{H}$ to be of lower semi-continuous nature provided that the Niemytzki operator corresponding to $\mathcal{H}$ is of lower semi-continuous nature and has non-empty, closed, and decomposable values.

Theorem 4.4. *[15]-[31],. Consider $\hbar$ represent a separable space equipped with a metric $\varnothing : \hbar \to \theta\left(\mathcal{L}^1([0, \dot{\psi}], \mathsf{B})\right)$ is a set-valued operator. Suppose $\varnothing$ provides the ability for continuous selection, means, there is a continuous operator $\bar{\kappa} : \hbar \to \theta\left(\mathcal{L}^1([0, \dot{\psi}], \mathsf{B})\right)$ in such a way that $\bar{\kappa}(\hbar) \in \varnothing(\hbar)$.*

(A10) *$\mathcal{H} : [0, \dot{\psi}] \times \mathsf{B} \to \theta(\mathsf{B})$ is a set-valued mapping in such a way that:*

- *$(\mathsf{t}, u) \mapsto \mathcal{H}(\mathsf{t}, \hbar)$ is $\mathcal{L} \otimes \mathcal{B}$-measurable;*
- *$\hbar \mapsto \mathcal{H}(\mathsf{t}, \hbar)$ is equipped with lower semi-continuous for $\mathsf{t} \in [0, \dot{\psi}]$.*

(A11) *To given every $q > 0$, there is a mapping $\bar{\kappa}_q \in \mathcal{L}^1([0, \dot{\psi}], \mathsf{B})$ in a way that $\|\mathcal{H}(\mathsf{t}, \hbar)\|_p \leq \bar{\kappa}_q(\mathsf{t})$ for $\mathsf{t} \in [0, \dot{\psi}]$ together with $\hbar \in \mathsf{B}$ and $|\hbar| \leq q$.*

Theorem 4.5. *let us assume that conditions $(A2) - (A4), (A6), (A10)$, together with $(A11)$ are fulfilled. Then equation (1.1) contains at least one solution.*

Proof. The hypotheses (A10) including with (A11) relative to Lemma (4.6), $\mathcal{H}$ is in a way lower semi continuous. From Theorem (4.5), there is a continuous mapping $f : C([0, \dot{\psi}], \mathsf{B}) \to \mathcal{L}^1([0, \dot{\psi}], \mathsf{B})$ in a way that $f(\hbar) \in \mathcal{H}(\hbar)$ for all $\hbar \in C([0, \dot{\psi}], \mathsf{B})$. Let us suppose the problem:

Obviously, $\hbar \in AC^1([0, \dot{\psi}], \mathsf{B})$ addresses this issue, then $\hbar$ is a answer of equation (1.1). Reformulating this problem as a fixed-point problem through the mapping $\varnothing_1 : C([0, \dot{\psi}], \mathsf{B}) \to C([0, \dot{\psi}], \mathsf{B})$ expressed as

$$\varnothing_1(\hbar)(\mathsf{t}) = P_\hbar(\mathsf{t}) + \int_0^{\dot{\psi}} \varpi(\mathsf{t}, s) f(\hbar)(s) ds$$

through which the mappings $P_\hbar$ and ϖ are expressed by equations (3.13) and (3.3), respectively. Using conditions $(A2) - (A4)$ and $(A6)$, We can readily demonstrate that the map $\varnothing_1$ fulfills all the requirements of the Leray-Schauder alternative. □

5. Topological framework

We now establish the topological framework containing the solution set given by equation (1.1).

Theorem 5.1. *Consider Hypothesis (A1) along with upcoming additional conditions:*
(A12) *There is a function $p \in \Delta(\jmath, \mathsf{B}^+)$ in a way that $\|\mathcal{H}(\mathsf{t}, u)\|_p \leq p(\mathsf{t})(|u|+1)$ to every $\mathsf{t} \in \jmath$ and $u \in \mathsf{B}$.*
(A13) *A function $p_1 \in \Delta(\jmath, \mathsf{B}^+)$ in a way that $\|\kappa(\mathsf{t}, u)\| \leq p_1(\mathsf{t})(|u|+1)$ to every $\mathsf{t} \in \jmath$ and $u \in \mathsf{B}$.*
(A14) *A function $p_2 \in \Delta(\jmath, \mathsf{B}^+)$ in a way that $\|\bar{\kappa}(\mathsf{t}, u)\| \leq p_2(\mathsf{t})(|u|+1)$ to every $\mathsf{t} \in \jmath$ and $u \in \mathsf{B}$,*
together with

$$\tfrac{\dot{\psi}(\dot{\psi}+1)}{\dot{\psi}+2}\tfrac{\wp+1}{\wp}\left[\|p_1\|_\infty + \|p_2\|_\infty + \bar{\varpi}\tfrac{\dot{\psi}+2}{\dot{\psi}(\dot{\psi}+1)}\|p\|_{\mathcal{L}^\infty}\right] < 1$$

is satisfied, then the answer set of equation (1.1) *is non-empty and closed as well as bounded in $\Delta(\jmath, \mathsf{B})$.*

Proof. Assume $S = \{\hbar \in \Delta(\jmath, \mathsf{B}) : \hbar$ is a solution of (1.1) $\}$. By Theorem (3.6), it is well understood that $S \neq \emptyset$. Our goal is to demonstrate that S is compact.

Suppose a sequence $(\hbar_n)_{n\in\varnothing} \subset S$; for each n, there is $v_n \in S_{F,\hbar_n}$ in such a way that, for any $\mathsf{t} \in \jmath$,

$$\hbar_n(\mathsf{t}) = P_{\hbar_n}(\mathsf{t}) + \textstyle\int_0^{\dot{\psi}} \varpi(\mathsf{t}, s)v_n(s)ds$$

meanwhile

$$P_{\hbar}(\mathsf{t}) = \tfrac{\dot{\psi}+1-\mathsf{t}}{\dot{\psi}+2}\textstyle\int_0^{\mathsf{t}} \kappa\left(s, \hbar_n(s)\right) ds + \tfrac{\dot{\psi}+1}{\dot{\psi}+2}\textstyle\int_0^{\mathsf{t}} \bar{\kappa}\left(s, \hbar_n(s)\right) ds$$

and $\varpi(\mathsf{t}, s)$ is defined by equation (3.3).
By conditions (A12) to (A14), there is a constant $\wp_1 > 0$ in such a way that

$$\|\hbar_n\|_\infty \leq \wp_1 \quad \text{for every } n \geq 1$$

As shown in Step 3 of Theorem (3.6), the expression $\{\hbar_n : n \geq 1\}$ is equicontinuous in $\Delta(\jmath, \mathsf{B})$. Based on the Arzela-Ascoli Theorem, there is a subsequence $\{\hbar_{n_m}\}$ that converges to $\hbar$ in $\Delta(\jmath, \mathsf{B})$.

Next, we demonstrate the existence of $v(\cdot) \in H(\cdot, \hbar(\cdot))$ in a way that

$$\hbar(\tau) = P_{\hbar}(\mathsf{t}) + \textstyle\int_0^{\dot{\psi}} \varpi(\mathsf{t}, s)v(s)ds$$

Due to $H(\mathsf{t}, \cdot)$ contains upper semi-continuity, for any $\epsilon > 0$, there is $n_0(\epsilon) \geq 0$ in a way that, for every $n \geq n_0$,

$$v_n(\mathsf{t}) \in \mathcal{H}\left(\mathsf{t}, \hbar_n(\mathsf{t})\right) \subset \mathcal{H}(\mathsf{t}, \hbar(\mathsf{t})) + \epsilon B(0,1) \quad \text{for all } \mathsf{t} \in \jmath$$

Because $\mathcal{H}(\cdot, \cdot)$ has closed and bounded values, there is a subsequence $v_{n_m}(\cdot)$ in a way that

$$v_{n_m}(\cdot) \to v(\cdot) \quad \text{as } m \to \infty$$

and

$$v(\mathsf{t}) \in \mathcal{H}(\mathsf{t}, \hbar(\mathsf{t})) \quad \text{for all } \mathsf{t} \in \jmath$$

It results that the subsequence $v_{n_m}(\mathsf{t})$ is integrably constrained. By leveraging the Lebesgue Dominated Convergence Theorem, we deduce that v is an element of $\mathcal{L}^1(\jmath, \mathsf{B})$, thereby confirming that v belongs to $S_{\mathcal{H},\hbar}$. Thus,

$$\hbar(\mathsf{t}) = P_\hbar(\mathsf{t}) + \int_0^{\dot\psi} \varpi(\mathsf{t}, s)v(s)ds$$

Therefore, S is compact in $\Delta(\jmath, \mathsf{B})$. □

6. Application

Present segment demonstrate the practical applicability of the theoretical findings for the problem described by equation (1.1). Examine the following boundary value problem :

(6.1)

$${}^cD^{\frac{3}{2}}\hbar(\mathsf{t}) \in \left\{ -\hbar^3(\mathsf{t}) + \cos(\hbar(\mathsf{t}-\kappa)) + \int_0^t \exp(\mathsf{t}-s)\hbar(s)ds \right\}, \mathsf{t} \in [0,1]$$

Associated with boundary conditions:

$$\hbar(0) - \hbar'(0) = \int_0^1 s^5(1+|\hbar(s)|) \tag{6.2}$$

$$\hbar(1) + \hbar'(1) = \int_0^1 s^5(1+|\hbar(s)|)ds \tag{6.3}$$

We define the set:

$$-\hbar^3(\mathsf{t}) + \cos(\hbar(\mathsf{t}-\kappa)) + \int_0^t \exp(\mathsf{t}-s)\hbar(s)ds = \{v \in \mathsf{B} : f_1 \le v \le f_2\}$$

where $f_1, f_2 : \jmath \times \mathsf{B} \times \mathsf{B} \times \mathsf{B} \to \mathsf{B}$ are measurable with respect to t and satisfy the Lipschitz condition in $\hbar$. It is assumed that for each $\mathsf{t} \in \jmath$, $f_1(\mathsf{t}, \cdot)$ is lower semi-continuous, and $f_2(\mathsf{t}, \cdot)$ is upper semi-continuous. Furthermore, suppose:

$$\max\left(|f_1(\mathsf{t},\hbar)|, |f_2(\mathsf{t},\hbar)|\right) \le \tfrac{\mathsf{t}}{9}(1+|\hbar|), \qquad \text{for all } \mathsf{t} \in \jmath \text{ and } \hbar \in \mathsf{B}$$

From equation (3.3), the function ϖ is given by:

$$\varpi(\mathsf{t}, s) = \begin{cases} \frac{(\mathsf{t}-s)^{\delta-1}}{\Gamma(\delta)} - \frac{(1+\mathsf{t})(\mathsf{t}-s)^{\delta-1}}{3\Gamma(\delta)} - \frac{(1+\mathsf{t})(1-s)^{\delta-2}}{3\Gamma(\delta-1)}, & 0 \le s \le \mathsf{t} \\ -\frac{(1+\mathsf{t})(\mathsf{t}-s)^{\delta-1}}{3\Gamma(\delta)} - \frac{(1+\mathsf{t})(1-s)^{\delta-2}}{3\Gamma(\delta-1)}, & \mathsf{t} \le s < 1 \end{cases}$$

where $\delta = \frac{3}{2}$.

We take $\dot\psi = 1, \phi_\kappa(\mathsf{t}) = \mathsf{t}^5, \phi_{\bar\kappa}(\mathsf{t}) = \mathsf{t}^5$, and the constants $a = \frac{1}{9}, b = \frac{1}{9}, c = \frac{1}{9}$. Additionally, the functions $\dot\psi(\hbar) = 1+\hbar, \dot\psi^*(\hbar) = 1+\hbar$, and $\bar{\dot\psi}(\hbar) = 1+\hbar$ for all $\hbar \in [0,\infty)$. Now, let us compute:

$$\int_0^1 \varpi(\mathsf{t}, s)ds = \int_0^{\mathsf{t}} \varpi(\mathsf{t}, s)ds + \int_{\mathsf{t}}^1 \varpi(\mathsf{t}, s)ds$$

This evaluates to:

$$\frac{\mathsf{t}^\delta}{\Gamma(\delta+1)} + \frac{(1+\mathsf{t})(1-\mathsf{t})^\delta}{3\Gamma(\delta+1)} - \frac{(1+\mathsf{t})}{3\Gamma(\delta+1)} + \frac{(1+\mathsf{t})(1-\mathsf{t})^{\delta-1}}{3\Gamma(\delta)}$$
$$-\frac{(1+\mathsf{t})}{3\Gamma(\delta)} - \frac{(1+\mathsf{t})(1-\mathsf{t})^\delta}{3\Gamma(\delta+1)} - \frac{(1+\mathsf{t})(1-\mathsf{tt})^{\delta-1}}{3\Gamma(\delta)}$$

From these calculations, we deduce:

$$\bar\varpi < \tfrac{3}{\Gamma(\delta+1)} + \tfrac{1}{\Gamma(\delta)} < 5$$

This implies that when $M > 7/2$, the condition (3.13) is satisfied. Furthermore, the map H is upper semi-continuous, convex-valued, and compact. Hence, the BVP (6.1)-(6.3) has at least one solution x on $\jmath$, as all the requirements of Theorem (3.6) are fulfilled.

Conclusion

The manuscript, successfully executed the Leray-Schauder fixed point theorem to demonstrate the solution's existence for boundary value problems involving non-linear fractional integrodifferential inclusions with discrete delay and integral conditions. The use of this theorem provides a robust framework for ensuring the solution's existence for convex as well as non-convex settings, offering flexibility in addressing complex mathematical models. The detailed approach presented here enhances our understanding of how such boundary value problems can be systematically approached, laying the groundwork for future applications in real-world systems that rely on memory effects, such as control systems, population dynamics, and engineering processes. This work contributes significantly to the growing field of fractional differential equations, showcasing their potential in modelling dynamic systems with cumulative and delayed effects.

Future research could focus on exploring the uniqueness and stability of these solutions, particularly in non-convex scenarios, as well as developing efficient numerical methods for practical implementation in real-time applications. Investigating the extension of these models to multidimensional systems or incorporating random effects could further enhance their applicability in areas such as climate modelling, economics, and financial forecasting, where delayed and cumulative responses are critical. This study not only provides a theoretical foundation but also opens new avenues for deeper exploration and practical advancements in various scientific and engineering disciplines.

Future scope

The existence results for boundary value problems involving non-linear fractional integro-differential inclusions with delays contains exciting possibilities for control theory. These results can be applied to controllability, ensuring systems can be driven to desired states under fractional dynamics and delays. They also provide a foundation for optimal control problems, helping to find strategies that minimize costs in systems with memory effects. Future research could explore computational methods to solve these problems in real-world systems, such as engineering and biology. Drawing on related studies would help expand these findings to broader applications. This could lead to more practical and efficient complex dynamic systems control strategies.

References

[1] Alil, F., Amara A., Zennir K., and Radwan, T., On Hybrid and Non-Hybrid Discrete Fractional Difference Inclusion Problems for the Elastic Beam Equation. Fractal and Fractional, 8(8), 486, 2024.

[2] Abdeljawad T. On Riemann and Caputo fractional differences. Computers and Mathematics with Applications, 62(3), 1602-1611, 2011).

[3] Agarwal R. P., Benchohra M., and Hamani, S. Boundary value problems for fractional differential equations, 2009.

[4] Benchohra M., and Hamani S. Non-linear boundary value problems for differential inclusions with Caputo fractional derivative, 2008.

[5] Benchohra M., Henderson J., Ntouyas S. K., and Ouahab A. Existence results for fractional order functional differential equations with infinite delay. Journal of Mathematical Analysis and Applications, 338(2), 1340-1350, 2008.

[6] Colombo A. Bressan G. Extensions and selections of maps with decomposable values. Studia math 90, 69-86, 1988.

[7] Covitz H., and Nadler S. B. Multi-valued contraction mappings in generalized metric spaces. Israel Journal of Mathematics, 8, 5-11, 1970.

[8] Dholkawala Z. F., Sarma H. K., and Kam S. I. Application of fractional flow theory to foams in porous media. Journal of Petroleum Science and Engineering, 57(1-2), 152-165, 2007.

[9] Dineshkumar C., Sooppy Nisar K., Udhayakumar R., and Vijayakumar V. A discussion on approximate controllability of Sobolev-type Hilfer neutral fractional stochastic differential inclusions. Asian Journal of Control, 24(5), 2378-2394, 2022.

[10] Das S., and Das S. Application of generalized fractional calculus in other science and engineering fields. Functional Fractional Calculus, 437-492, 2011.

[11] Ekeland I. C. Castaing and M. Valadier, Convex analysis and measurable multifunctions. 950-956, 1978.

[12] Engheta N. On fractional calculus and fractional multipoles in electromagnetism. IEEE Transactions on Antennas and Propagation, 44(4), 554-566, 1996.

[13] Farman M., Ahmad A., Zehra A., Nisar K. S., Hincal E., and Akgul A. Analysis and controllability of diabetes model for experimental data by using fractional operator. Mathematics and Computers in Simulation, 218, 133-148, 2024.

[14] Guide, A. H. S., Infinite dimensional analysis. New York: Springer, 2006.

[15] Hamani S., Benchohra M., and Graef J. R., Existence results for boundary-value problems with nonlinear fractional differential inclusions and integral conditions. Electronic Journal of Differential Equations (EJDE), 2010.

[16] I. Podlubny, Fractional Differential Equation, Academic Press, San Diego, 1999.

[17] Kilbas, Anatolii Aleksandrovich, Hari M. Srivastava, and Juan J. Trujillo. Theory and applications of fractional differential equations. Vol. 204. elsevier, 2006.

[18] Kilbas, Anatoly A., and Sergei A. Marzan. Nonlinear differential equations with the Caputo fractional derivative in the space of continuously differentiable functions. Differential Equations 41 (2005): 84-89.

[19] Kelly C. K., Bowler M. G., Joy J. B., and Williams J. N. Fractional abundance and the ecology of community structure. arXiv preprint arXiv:1008.2527, 2010.

[20] Kaufmann E., and Ebene Mboumi. Positive solutions of a boundary value problem for a non-linear fractional differential equation. Electronic

Journal of Qualitative Theory of Differential Equations 3, 1-11, 2008.
[21] K. S. Miller and B. Ross, An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993.
[22] Kisielewicz Michal. Differential inclusions and optimal control. (No Title) 1991.
[23] Lamy, Xavier, Andrew Lorent, and Guanying Peng. Quantitative rigidity of differential inclusions in two dimensions. International Mathematics Research Notices 2024.8, 6325-6349, 2024.
[24] Ma Y. K., Vijayakumar V., Shukla A., Nisar K. S., Thilagavathi K., Nashine H. K., and Zakarya M. Discussion on the existence of mild solution for fractional derivative by Mittag-Leffler kernel to fractional stochastic neutral differential inclusions. Alexandria Engineering Journal, 63, 271-282, 2023.
[25] Meral F. C., Royston T. J., and Magin R. Fractional calculus in viscoelasticity: an experimental study. Communications in nonlinear science and numerical simulation, 15(4), 939-945, 2010.
[26] Momani S. M., and S. B. Hadid. Some comparison results for integro-fractional differential inequalities. J. Fract. Calc 24, 37-44, 2003.
[27] Muslim M.; Kumar A., Feckan M. Existence, uniqueness and stability of solutions to second order non-linear differential equations with non-instantaneous impulses. J. King Saud Univ. Sci. 2016.
[28] Ouahab, Abdelghani. Some results for fractional boundary value problem of differential inclusions. Non-linear Analysis: Theory, Methods and Applications 69.11, 3877-3896, 2008.
[29] Podlubny Igor. Geometric and physical interpretation of fractional integration and fractional differentiation. arXiv preprint math/0110241 2001.
[30] Rihan F. A. Numerical modeling of fractional-order biological systems. In Abstract and applied analysis (Vol. 2013, No. 1, p. 816803). Hindawi Publishing Corporation, 2013.
[31] Rockafellar, R. T., Monotone operators and the proximal point algorithm. SIAM journal on control and optimization, 14(5), 877-898, 1976.
[32] Tian Y., He G., Liu Z., Zhong L., Yang X., Stanley H. E., and Tu Z. The impact of memory effect on resonance behavior in a fractional oscillator with small time delay. Physica A: Statistical Mechanics and its Applications, 563, 125383, 2021.
[33] Ulam, S.M. A Collection of the Mathematical Problems; Interscience: New York, NY, USA, 1960.
[34] Williams W. K., Vijayakumar V., Nisar K. S., and Shukla A. Atangana-Baleanu semi-linear fractional differential inclusions with infinite delay: existence and approximate controllability. Journal of Computational and Non-linear Dynamics, 18(2), 021005, 2023.

School of Advanced Sciences, Department of Mathematics, VIT-AP University, Amaravati 522237, India
Email address: brijendrao60@gmail.com*

School of Advanced Sciences, Department of Mathematics, VIT-AP University, Amaravati 522237, India
Email address: soni.iitkgp@gmail.com